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Vorticity and curvature at a stream surface

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Abstract

If S is a stream surface in a flow, we show the relationship between the three components of the vorticity field on S and the curvatures of the streamlines (geodesic torsion, normal curvature and geodesic curvature).

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1. Introduction

The study of the vorticity on the interfacial surfaces is of considerable interest. Recent works relate the geometry of the surfaces with the components of the vorticity, for example Wu [1], and Dopazo, Lozano and Barreras [2]. If S is a steady and free surface (both tangential components of the stress tensor vanish at the surface), Longuet-Higgins [3] shows the relationship between the tangential components of the vorticity field on S and the normal curvatures of S. In this paper we consider the case where S is a stream surface, and we find the relationship between the three components of the vorticity field on S and the curvatures of the streamlines.

We consider a stream surface S in a flow; that is: let S be a smooth surface of $\mathbb{R}^3$ (oriented Euclidean three-dimensional space) tangent to the smooth velocity vector field $\vec{u} = \vec{u}(p,t)$ of the flow at any fixed time t, which moves with time. Let $\vec{x}$ be a parametrization of the smooth surface S:

$$\begin{aligned}
&\vec{x} : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3,\\
&\quad (\xi_1, \xi_2) \longrightarrow \vec{x}(\xi_1, \xi_2) = \big(x(\xi_1, \xi_2), y(\xi_1, \xi_2), z(\xi_1, \xi_2)\big).
\end{aligned} \tag{1}$$

Let $\vec{N}(\xi_1, \xi_2)$ be the normal vector given by the parametrization

$$\vec{N} = \frac{\partial\vec{x}/\partial\xi_1 \times \partial\vec{x}/\partial\xi_2}{\|\partial\vec{x}/\partial\xi_1 \times \partial\vec{x}/\partial\xi_2\|}.$$

We consider the velocity vector field $\vec{u} = \vec{u}(p,t)$ and the vorticity vector field $\vec{\omega}(p,t) = \operatorname{curl}(\vec{u})$ on the stream surface S. Let $\vec{u}^{\perp}$ be the tangent vector field on S such that $\{\frac{\vec{u}}{q}, \frac{\vec{u}^{\perp}}{q}, \vec{N}\}$ is an orthonormal, direct basis (positively

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oriented), where the velocity of the particle $q = \|\vec{u}\| \neq 0$. Let $\omega_\parallel$, $\omega_\perp$ be the tangent components on S of the vorticity: $\omega_\parallel$ is the component parallel to the flow and $\omega_\perp$ is the perpendicular component to the flow in the direction of $\vec{u}^\perp$. Let ω_3 be the vertical component of the vorticity to S.

With these notations, in this work we will find the following

Formulation. *Let S be a stream surface in a flow. Let p be a point of a streamline on S with $q \neq 0$. Then we have the following equations*

$$\begin{aligned}
\omega_\parallel &= -2q\tau_g - 2D_p\left(\frac{\vec{u}^\perp}{q}, \vec{N}\right),\\
\omega_\perp &= -2qk_n + 2D_p\left(\frac{\vec{u}}{q}, \vec{N}\right),\\
\omega_3 &= 2qk_g - 2D_p\left(\frac{\vec{u}}{q}, \frac{\vec{u}^\perp}{q}\right),
\end{aligned} \tag{2}$$

where τ_g, k_n and k_g are, respectively, the geodesic torsion, the normal curvature and the geodesic curvature of the streamline, and D_p is the 2-covariant rate-of-strain tensor at p.

2. Geometric preliminaries

This section is dedicated to the readers that are not familiarized with the Differential Geometry.

2.1. 2-covariant rate-of-strain tensor field on S

The concept of the rate-of-strain tensor is very well-known but we need a review with algebraic language.

Given the velocity vector field $\vec{u}$ on the space $\mathbb{R}^3$, we have the well-know 2-covariant rate-of-strain tensor field D; we will denote by e_{ij} the components of D on the affine frame $\{p; \vec{a}_1, \vec{a}_2, \vec{a}_3\}$. We can construct for each point p the tensor (the bilinear application):

$$\begin{aligned}
T_p(\mathbb{R}^3) \times T_p(\mathbb{R}^3) &\xrightarrow{D_p} \mathbb{R},\\
(\vec{v}, \vec{w}) &\longrightarrow D_p(\vec{v}, \vec{w}),
\end{aligned} \tag{3}$$

where the values on the couples of the basis $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ are

$$D_p(\vec{a}_i, \vec{a}_j) = e_{ij}. \tag{4}$$

Therefore the value on any couple $\vec{v}$, $\vec{w}$ of vectors at the point p with $\vec{v} = v_1\vec{a}_1 + v_2\vec{a}_2 + v_3\vec{a}_3$, $\vec{w} = w_1\vec{a}_1 + w_2\vec{a}_2 + w_3\vec{a}_3$ is

$$D_p(\vec{v}, \vec{w}) = (v_1, v_2, v_3)\begin{pmatrix} e_{11} & e_{12} & e_{13}\\ e_{21} & e_{22} & e_{23}\\ e_{31} & e_{32} & e_{33}\end{pmatrix}_p \begin{pmatrix} w_1\\ w_2\\ w_3\end{pmatrix} = \vec{v} D_p \vec{w}^t \in \mathbb{R}. \tag{5}$$

2.2. Geodesic curvature and geodesic torsion

To define the geodesic curvature k_g of the curve $\gamma(r) \subset S$ at a point p we consider, E_p, the tangent plane to S at p, we project $\gamma(r)$ onto E_p following the direction $\vec{N}(p)$, and we obtain the plane curve $\bar{\gamma}(r)$. The curvature of $\bar{\gamma}(r)$ at p is the geodesic curvature k_g. The sign of k_g is positive or negative according to the basis $\{\bar{\gamma}'(0), \bar{\gamma}''(0), \vec{N}(p)\}$ is direct or not.

We will define the geodesic torsion τ_g of the line C on the surface S: let $\gamma(s) = \vec{x}(\xi_1(s), \xi_2(s))$ be a parametrization of C by arc length so that $p = \gamma(0)$. Let $\vec{G}(\gamma(s)) = \vec{G}(s)$ be the unit vector such that $\{\vec{T}(s), \vec{G}(s), \vec{N}(s)\}$ is an orthonormal, direct basis with $\frac{\mathrm{d}}{\mathrm{d}s}\gamma(s) = \vec{T}(\gamma(s)) = \vec{T}(s)$. The geodesic torsion, $\tau_g(p)$, of C at p is defined by

$$\tau_g = \frac{\mathrm{d}\vec{N}}{\mathrm{d}s}(\gamma(0)) \cdot \vec{G}(\gamma(0)), \tag{6}$$

where $\cdot$ denotes the scalar product.

The affine frame $\{\gamma(s); \vec{T}(s), \vec{G}(s), \vec{N}(s)\}$ is called Darboux–Ribaucour trihedron and we have

$$\begin{aligned}
\frac{\mathrm{d}\vec{T}}{\mathrm{d}s} &= k_g\vec{G} + k_n\vec{N},\\
\frac{\mathrm{d}\vec{G}}{\mathrm{d}s} &= -k_g\vec{T} - \tau_g\vec{N},\\
\frac{\mathrm{d}\vec{N}}{\mathrm{d}s} &= -k_n\vec{T} + \tau_g\vec{G},
\end{aligned}\tag{7}$$

where k_n is the normal curvature of the curve γ.

Warning. Other geometers define the geodesic torsion with the opposed sign: $(\tau_g = -\frac{\mathrm{d}\vec{N}}{\mathrm{d}s}\cdot\vec{G} \to \frac{\mathrm{d}\vec{N}}{\mathrm{d}s} = -k_n\vec{T} - \tau_g\vec{G})$.

For this section see, for example, Do Carmo [4] or other books of Differential Geometry of curves and surfaces.

3. Components of the vorticity on the stream surface S

Let S be a stream surface in a flow. Let p be a point of a streamline on S with $q \neq 0$. Let k_n, τ_g and k_g be, respectively, the geodesic torsion, the normal curvature and the geodesic curvature of the streamline. Using the same ideas of Longuet-Higgins [3] and with the algebraic language of the rate-of-strain tensor D_p, we obtain the following equations:

$$\begin{aligned}
\omega_\parallel &= -2q\tau_g - 2D_p\left(\frac{\vec{u}^\perp}{q}, \vec{N}\right),\\
\omega_\perp &= -2qk_n + 2D_p\left(\frac{\vec{u}}{q}, \vec{N}\right).
\end{aligned}\tag{8}$$

To find the formula for the vertical component ω_3, we have used the geometric interpretations of the vorticity and the rate-of-strain tensor (see, for example, Aris [5]). We obtain the following equation:

$$\omega_3 = 2qk_g - 2D_p\left(\frac{\vec{u}}{q}, \frac{\vec{u}^\perp}{q}\right).\tag{9}$$

But, we can demonstrate Eqs. (8) and (9) in a shorter and compact way. Then we give here this proof, which was proposed by Reviewer #2 of this paper:

Result. *Let $\vec{u}$ be a smooth vector field in a domain Ω of the oriented Euclidean three-dimensional space. Let S be a smooth surface in Ω, such that at each point $a \in S$, the vector $\vec{u}(a)$ is tangent to S at a. Let $\vec{x}$ be a parametrization of the surface S. Let $\vec{\omega}$ and D be, respectively, the vorticity field and the rate-of-strain tensor field of the vector field $\vec{u}$. Let p be a point in S such that $q = \|\vec{u}(p)\| \neq 0$. Let $\gamma : s \to \gamma(s)$ be the integral curve of the vector field $\vec{u}$, which satisfies $\gamma(0) = p$. We know that, at least for $|s|$ small enough, $\gamma(s)$ lies in S. Let $\omega_\parallel$, $\omega_\perp$, and ω_3 the components of $\vec{\omega}(p)$ in the orthonormal, direct basis $\{\vec{e}_\parallel, \vec{e}_\perp, \vec{N}(p)\}$, where $\vec{e}_\parallel = \frac{\vec{u}(p)}{q}$ and $\vec{N}(p)$ is the normal to S at p given by $\vec{x}$. Then*

$$\begin{aligned}
\omega_\parallel &= -2q\tau_g - 2D_p(\vec{e}_\perp, \vec{N}),\\
\omega_\perp &= -2qk_n + 2D_p(\vec{e}_\parallel, \vec{N}),\\
\omega_3 &= 2qk_g - 2D_p(\vec{e}_\parallel, \vec{e}_\perp),
\end{aligned}\tag{10}$$

where τ_g, k_n and k_g are, respectively, the geodesic torsion, the normal curvature and the geodesic curvature of the curve γ at p.

Proof. To shorten the notations, we will set

$$\vec{e}_\parallel = \vec{e}_1, \quad \vec{e}_\perp = \vec{e}_2, \quad \vec{N}(p) = \vec{e}_3,\tag{11}$$

and we will denote by x, y and z the coordinates in the affine frame $\{p; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$. We will denote by u_1, u_2, u_3 and ω_1, ω_2, ω_3 the three components of the vector fields $\vec{u}$ and $\vec{\omega}$ in the orthonormal basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Similarly, we will set, for all i and j $(1 \leqslant i, j \leqslant 3)$:

$$e_{ij} = D_p(\vec{e}_i, \vec{e}_j). \tag{12}$$

Then Eqs. (10) can be written as

$$\begin{aligned} \omega_1(p) &= -2q\tau_g - 2e_{23}, \\ \omega_2(p) &= -2qk_n + 2e_{13}, \\ \omega_3(p) &= 2qk_g - 2e_{12}. \end{aligned} \tag{13}$$

But according to the expressions of $\vec{\omega}$ and D, in an orthonormal affine frame, in terms of the partials derivatives of the components of the vector field $\vec{u}$, we have

$$\begin{aligned} &\omega_1 = \frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \quad \omega_2 = \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \quad \omega_3 = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}, \\ &e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\bigg|_{x=y=z=0} \quad \text{with } x_1 = x, x_2 = y \text{ and } x_3 = z. \end{aligned} \tag{14}$$

Therefore we have

$$\begin{aligned} \omega_1(p) + 2e_{23} &= 2\frac{\partial u_3}{\partial y}\bigg|_{x=y=z=0}, \\ \omega_2(p) - 2e_{13} &= -2\frac{\partial u_3}{\partial x}\bigg|_{x=y=z=0}, \\ \omega_3(p) + 2e_{12} &= 2\frac{\partial u_2}{\partial x}\bigg|_{x=y=z=0}. \end{aligned} \tag{15}$$

Therefore Eqs. (10) are equivalent to

$$\begin{aligned} -q\tau_g &= \frac{\partial u_3}{\partial y}\bigg|_{x=y=z=0} \quad \text{(see warning of Subsection 2.2)}, \\ qk_n &= \frac{\partial u_3}{\partial x}\bigg|_{x=y=z=0}, \\ qk_g &= \frac{\partial u_2}{\partial x}\bigg|_{x=y=z=0}. \end{aligned} \tag{16}$$

Thus it remains to prove Eqs. (16).

Let us assume that the curve $s \mapsto \gamma(s)$ is parametrized by the arc length s, with $\gamma(0) = p$, and we consider the variation of its Darboux–Ribaucour trihedron (Eqs. (7)). The vector $\vec{T} = \vec{T}(\gamma(s))$ can be defined not only on the curve γ, but at any point where $\vec{u}$ does not vanish, since we have

$$\vec{T} = \frac{\vec{u}}{\|\vec{u}\|}. \tag{17}$$

Therefore we may write

$$\frac{d\vec{T}}{ds} = \frac{\partial \vec{T}}{\partial x}\frac{dx}{ds} + \frac{\partial \vec{T}}{\partial y}\frac{dy}{ds} + \frac{\partial \vec{T}}{\partial z}\frac{dz}{ds}, \tag{18}$$

where x, y, and z stand for $x(\gamma(s))$, $y(\gamma(s))$, and $z(\gamma(s))$, respectively. But for $s = 0$, we have

$$\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)\bigg|_{s=0} = (1, 0, 0). \tag{19}$$

So we have

$$\left.\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right|_{s=0} = \left.\frac{\partial \vec{T}(x,y,z)}{\partial x}\right|_{x=y=z=0} = \frac{1}{\|\vec{u}(p)\|}\left.\frac{\partial \vec{u}(x,y,z)}{\partial x}\right|_{x=y=z=0} - \frac{1}{\|\vec{u}(p)\|^2}\left.\frac{\partial \|\vec{u}(x,y,z)\|}{\partial x}\right|_{x=y=z=0}\vec{u}(p)$$
$$= \frac{1}{q}\left.\frac{\partial \vec{u}(x,y,z)}{\partial x}\right|_{x=y=z=0} - \frac{1}{q}\left.\frac{\partial \|\vec{u}(x,y,z)\|}{\partial x}\right|_{x=y=z=0}\vec{e}_1. \tag{20}$$

Since $\vec{e}_1 \cdot \vec{e}_2 = \vec{e}_1 \cdot \vec{e}_3 = 0$, by making the scalar product of both sides of the above equation by $\vec{e}_2$, then by $\vec{e}_3$, we get

$$\left.\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right|_{s=0} \cdot \vec{e}_2 = \frac{1}{q}\left.\frac{\partial u_2(x,y,z)}{\partial x}\right|_{x=y=z=0},$$
$$\left.\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right|_{s=0} \cdot \vec{e}_3 = \frac{1}{q}\left.\frac{\partial u_3(x,y,z)}{\partial x}\right|_{x=y=z=0}. \tag{21}$$

Let us now write the first equation (7) for $s = 0$, and let us make the scalar product of both its sides first by the vector $\vec{e}_2 = \vec{G}(\gamma(0))$, then by the vector $\vec{e}_3 = \vec{N}(\gamma(0))$. We obtain

$$k_g\big(\gamma(0)\big) = \left.\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right|_{s=0} \cdot \vec{e}_2, \qquad k_n\big(\gamma(0)\big) = \left.\frac{\mathrm{d}\vec{T}}{\mathrm{d}s}\right|_{s=0} \cdot \vec{e}_3. \tag{22}$$

The above result and Eqs. (21) show that the second and the third formulae (16) (which are equivalent to the second and the third formulae (10)) are proven.

We still have to prove the first formula (10), or equivalently the first formula (16). Let us write the third equation (7) for $s = 0$, and let us make the scalar product of both its sides by $\vec{e}_2 = \vec{G}(\gamma(0))$. We get

$$\tau_g\big(\gamma(0)\big) = \left.\frac{\mathrm{d}\vec{N}}{\mathrm{d}s}\right|_{s=0} \cdot \vec{e}_2. \tag{23}$$

In a neighbourhood of p, the surface S can be described by the equation

$$z = h(x,y), \tag{24}$$

where h is a smooth function which satisfies, since the coordinate plane $z = 0$ is tangent to S at point p (coordinates $x = y = z = 0$):

$$z(0,0) = 0, \qquad \left.\frac{\partial h(x,y)}{\partial x}\right|_{x=y=0} = \left.\frac{\partial h(x,y)}{\partial y}\right|_{x=y=0} = 0. \tag{25}$$

Observe that the vector $\vec{N}(\vec{x}(x,y)) = \vec{N}(x,y,h(x,y))$ is given by

$$\vec{N}\big(x,y,h(x,y)\big) = \frac{1}{\sqrt{(\partial h/\partial x)^2 + (\partial h/\partial y)^2 + 1}}\left(-\frac{\partial h}{\partial x}\vec{e}_1 - \frac{\partial h}{\partial y}\vec{e}_2 + \vec{e}_3\right). \tag{26}$$

An easy calculation (with (19) and (25)) then shows that Eq. (23) is equivalent to

$$\tau_g\big(\gamma(0)\big) = -\left.\frac{\partial^2 h(x,y)}{\partial x \partial y}\right|_{x=y=0}. \tag{27}$$

Let us now write that the vector field $\vec{u}$ is tangent to S at each of its points, i.e. that $\vec{u}(x,y,h(x,y)) \cdot \vec{N}(x,y,h(x,y)) = 0$: we must have, an every point of S,

$$\frac{\partial h}{\partial x}u_1 + \frac{\partial h}{\partial y}u_2 - u_3 = 0. \tag{28}$$

We take the derivative of the left side of above expression with respect to y ($z = h(x,y)$ being regarded as a function of x and y). Then we make $x = y = 0$. Many terms vanish and we get

$$\left.\frac{\partial u_3(x,y,z)}{\partial y}\right|_{x=y=z=0} = u_1\left.\frac{\partial^2 h(x,y)}{\partial x \partial y}\right|_{x=y=0} = -q\tau_g\big(\gamma(0)\big). \tag{29}$$

We have proven the first formula (16), which is equivalent to the first formula (10). □

Finally, we summarize this work with the formulation presented in the introduction of this paper: Eqs. (2).

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